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Lie algebraic approach to the coupled-mode oscillator

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Abstract. The coupling of electromagnetic fields for the construction of light amplifiers is described by a certain Hamiltonian which can be expressed as the sum of three differential operators. Following the method of Steinberg and Dattoli *et al*, the solution of Schrödinger's equation for the Hamiltonian of the system is constructed from the solutions of the corresponding equations for the components of the Hamiltonian.

1. Introduction

The construction of light amplifiers and frequency converters is based on the coupling of light waves in nonlinear dielectric crystals. In fact the structure of crystals has long utilized the phenomenon of light scattering from atoms or molecules having two energy levels. The frequency of the incident beam may be shifted, up or down, by an amount equal to the difference in the two energy levels [1]. The resulting lower and higher frequency scattered waves are the Stokes and anti-Stokes components, respectively. In Brillouin scattering, for example, an intense monochromatic laser source induces parametric coupling between the two scattered electromagnetic fields and accoustical phonons in the scattering medium. We may therefore conclude that the problem of the frequency converter and the parametric amplifier, where three electromagnetic fields are coupled, has played a significant role in the field of quantum optics [2–5].

The most familiar Hamiltonian which describes three coupled modes is of the form

$$\frac{H}{\hbar} = \omega_1\left(a_1^+a_1 + \tfrac{1}{2}\right) + \omega_2\left(a_2^+a_2 + \tfrac{1}{2}\right) + \omega_3\left(a_3^+a_3 + \tfrac{1}{2}\right) + \gamma\left(a_1^+a_2a_3 + a_1a_2^+a_3^+\right) \tag{1.1}$$

where a_i, a_i^+ are the boson annihilation and creation operators of the ith mode of frequency ω_i, γ is the coupling constant, and $\hbar$ is Plank's constant. The energy conservation condition is expressed by the equality $\omega_1 = \omega_2 + \omega_3$. Under certain conditions, the mode a_1 can be replaced by its c-number, and then the Hamiltonian (1.1) takes the form

$$\frac{H}{\hbar} = \omega_2\left(a_2^+a_2 + \tfrac{1}{2}\right) + \omega_3\left(a_3^+a_3 + \tfrac{1}{2}\right) + \lambda\left(a_2a_3\mathrm{e}^{\mathrm{i}(\omega_1 t+\varphi(t))} + \text{ CC}\right). \tag{1.2}$$

Here $\varphi(t)$ is the phase pump which may be chosen so that $\varphi(t) = -\omega_1 t$, and $\lambda = \gamma|\alpha_1|$, where α_1 is the c-number for the operator a_1. In this case (1.2) can be re-written as

$$H = \omega(a^+a + b^+b + 1) + \lambda(ab + a^+b^+) \tag{1.3}$$

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where we have taken $a = a_2, b = a_3, \hbar = 1$ and the system is assumed to be at exact resonance.

It is interesting to point out that the evolution operator of the interaction part of the above Hamiltonian has been identified as a correlated squeeze operator, which is regarded as the $Su(1, 1)$-generalized coherent state, see for example [6, 7]. On the other hand one of us (MSA) added (with a different coupling parameter) the frequency converter part to (1.3), where the statistical aspect for such a system is considered, and the quantum mechanical treatment was also given. For more details see [8–10]. Our purpose in the present work is to employ a Lie algebraic approach in order to find the most general solution for the wavefunction in the Schrödinger picture.

The solution of the Cauchy problem for Schrödinger's equation

$$H\psi = \mathrm{i}\frac{\partial\psi}{\partial t} \qquad \psi = \tilde{\psi} \qquad \text{at} \quad t = 0 \tag{1.4}$$

lends itself to the methods of Lie algebra, as first laid down by Cartan and later developed by Miller [11]. This follows from the observation that the Hamiltonian operator (1.3) is a linear combination of the three operators a^+b^+, ab and $a^+a + b^+b + 1$ which are closed under the commutation relation. Thus they generate a three-dimensional Lie algebra to which the more recent techniques of Steinberg [12] and Dattoli *et al* [13] can be applied.

The underlying idea in Dattoli's approach is: given a partial differential equation, we first choose an appropriate Lie group G and identify a basis for its Lie algebra $L(G)$. Since G is isomorphic to a Lie matrix group G', the matrix realization of G is used to write the image of the partial differential equation in G'. This gives the expressions for the 1-parameter subgroups corresponding to the basis elements. Then, going back to the generators in G, we can use simple operational rules to arrive at the solution of the differential equation.

2. Associated differential equations

By defining

$$K_+ = a^+b^+ \qquad K_- = ab \qquad K_3 = \tfrac{1}{2}(a^+a + b^+b + 1) \tag{2.1}$$

we see that the Hamiltonian (1.3) becomes [14, 15]

$$H = 2\omega K_3 + \lambda(K_+ + K_-) \tag{2.2}$$

and that the commutation relations

$$[K_+, K_-] = -2K_3 \qquad [K_3, K_\pm] = \pm K_\pm \tag{2.3}$$

which can be recognized as an Su(1,1) Lie algebra, hold. Now we set

$$a = (2\omega)^{-1/2}(\omega q_1 + \mathrm{i}p_1) \qquad b = (2\omega)^{-1/2}(\omega q_2 + \mathrm{i}p_2) \tag{2.4}$$

where $p_1 = -\mathrm{i}\partial/\partial q_1$, $p_2 = -\mathrm{i}\partial/\partial q_2$ so that

$$[q_1, p_1] = [q_2, p_2] = \mathrm{i}. \tag{2.6}$$

We shall tackle the Cauchy problem (1.4) by first considering the three initial-value problems

$$-\mathrm{i}K_+ f = \partial f/\partial t \qquad f(q_1, q_2, 0) = \tilde{f}(q_1, q_2) \tag{2.7}$$

$$-\mathrm{i}K_- g = \partial g/\partial t \qquad g(q_1, q_2, 0) = \tilde{g}(q_1, q_2) \tag{2.8}$$

$$K_3 h = \partial h/\partial t \qquad h(q_1, q_2, 0) = \tilde{h}(q_1, q_2)\,. \tag{2.9}$$

Under the change of variables

$$q_1 = (2\omega)^{-1/2}(x+y) \qquad q_2 = (2\omega)^{-1/2}(x-y) \tag{2.10}$$

the differential equation in (2.7) becomes

$$4\mathrm{i}\frac{\partial f}{\partial t} = \left[\left(\frac{\partial^2}{\partial x^2} - 2x\frac{\partial}{\partial x} + x^2\right) - \left(\frac{\partial^2}{\partial y^2} - 2y\frac{\partial}{\partial y} + y^2\right)\right] f\,. \tag{2.11}$$

The general solution of (2.11), by separation of variables, is given by

$$f(x, y, t) = \sum_{m,n=-\infty}^{\infty} a_{mn} \exp\left[-\frac{\mathrm{i}}{4}(n^2 - m^2)t + \frac{1}{2}(x^2 + y^2) + \mathrm{i}(mx + ny)\right] \tag{2.12a}$$

where the constants a_{mn} are determined by the initial condition

$$\tilde{f}(x, y) = \exp\left[\tfrac{1}{2}(x^2 + y^2)\right] \sum_{m,n=-\infty}^{\infty} a_{mn} \exp[\mathrm{i}(mx + my)]\,. \tag{2.12b}$$

This equation clearly shows that a_{mn} are the Fourier coefficients of $\tilde{f}(x, y)\exp[-\frac{1}{2}(x^2+y^2)]$, given by

$$a_{mn} = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \tilde{f}(x, y)\exp\left[-\frac{1}{2}(x^2 + y^2)\right]\exp[\mathrm{i}(mx + ny)]\mathrm{d}x\mathrm{d}y\,. \tag{2.12c}$$

Using the inverse transformation

$$x = \sqrt{\frac{\omega}{2}}(q_1 + q_2) \qquad y = \sqrt{\frac{\omega}{2}}(q_1 - q_2) \tag{2.12d}$$

we can express the solution in terms of the physical coordinates (q_1, q_2):

$$f(q_1, q_2, t) = \sum_{m,n=-\infty}^{\infty} a_{mn} \exp\left[-\frac{\mathrm{i}}{4}(n^2 - m^2)t + \frac{\omega}{2}(q_1^2 + q_2^2) + \mathrm{i}\sqrt{\frac{\omega}{2}}\{(m+n)q_1 + (m-n)q_2\}\right] \tag{2.13}$$

where the constants a_{mn} are given by

$$a_{mn} = \frac{\omega}{4\pi^2}\int_{-\pi/\sqrt{2\omega}}^{\pi/\sqrt{2\omega}}\int_{-\pi/\sqrt{2\omega}}^{\pi/\sqrt{2\omega}} \tilde{f}(q_1, q_2)\exp\left[-\frac{\omega}{2}(q_1^2 + q_2^2)\right] \times \exp\left[\mathrm{i}\sqrt{\frac{\omega}{2}}((m+n)q_1 + (m-n)q_2)\right]\mathrm{d}q_1\mathrm{d}q_2\,. \tag{2.14}$$

The differential equation in (2.8) is

$$4\mathrm{i}\frac{\partial g}{\partial t} = \left[\left(\frac{\partial^2}{\partial x^2} + 2x\frac{\partial}{\partial x} + x^2\right) - \left(\frac{\partial^2}{\partial y^2} + 2y\frac{\partial}{\partial y} + y^2\right)\right] g \tag{2.15a}$$

and its general solution is

$$g(x, y, t) = \sum_{r,s=-\infty}^{\infty} b_{rs} \exp\left[-\frac{\mathrm{i}}{4}(s^2 - r^2)t - \frac{1}{2}(x^2 + y^2) + \mathrm{i}(rx + sy)\right] \tag{2.15b}$$

where

$$b_{rs} = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \tilde{g}(x, y) \exp\left[\frac{1}{2}(x^2 + y^2)\right] \exp[\mathrm{i}(rx + sy)]\mathrm{d}x\mathrm{d}y\,. \tag{2.15c}$$

Substituting back $x = \sqrt{\omega/2}(q_1 + q_2)$, $y = \sqrt{\omega/2}(q_1 - q_2)$ we get the solution of the problem (2.8) as

$$g(q_1, q_2, t) = \sum_{r,s=-\infty}^{\infty} b_{rs} \exp\left[\frac{-\mathrm{i}}{4}(s^2 - r^2)t - \frac{\omega}{2}(q_1^2 + q_2^2) + \mathrm{i}\sqrt{\frac{\omega}{2}}\{(r + s)q_1 + (r - s)q_2\}\right] \tag{2.16}$$

and

$$b_{rs} = \frac{\omega}{4\pi^2}\int_{-\pi/\sqrt{2\omega}}^{\pi/\sqrt{2\omega}}\int_{-\pi/\sqrt{2\omega}}^{\pi/\sqrt{2\omega}} \tilde{g}(q_1, q_2) \times \exp\left[\frac{\omega}{2}(q_1^2 + q_2^2)\right] \exp\left[\mathrm{i}\sqrt{\frac{\omega}{2}}((r + s)q_1 + (r - s)q_2)\right] \mathrm{d}q_1\mathrm{d}q_2\,. \tag{2.17}$$

Finally, (2.9) yields the differential equation

$$4\omega\frac{\partial h}{\partial t} = \left(\frac{\partial^2 h}{\partial q_1^2} - \omega^2 q_1^2 h\right) + \left(\frac{\partial^2 h}{\partial q_2^2} - \omega^2 q_2^2 h\right) \tag{2.18}$$

whose solution is

$$h(q_1, q_2, t) = \sum_{j,k=0}^{\infty} c_{jk} H_j(\sqrt{\omega}q_1) H_k(\sqrt{\omega}q_2) \exp\left[-\frac{\omega}{2}(q_1^2 + q_2^2) - \frac{1}{2}(j + k + 1)t\right] \tag{2.19a}$$

H_j being the Hermite polynomial of order j. At $t = 0$, we obtain

$$\tilde{h}(q_1, q_2) \exp\left[\tfrac{1}{2}\omega(q_1^2 + q_2^2)\right] = \sum_{j,k=0}^{\infty} c_{jk} H_j(\sqrt{\omega}q_1) H_k(\sqrt{\omega}q_2)\,. \tag{2.19b}$$

In view of the orthogonal relation

$$\int_{-\infty}^{\infty} H_j(\xi) H_k(\xi) \mathrm{e}^{-\xi^2} \mathrm{d}\xi = \begin{cases} 0 & j \neq k \\ 2^j j! \sqrt{\pi} & j = k \end{cases}$$

the constants c_{jk} in (2.19a) are given by

$$c_{jk} = \frac{\omega}{\pi}\frac{1}{2^{j+k} j! k!}\int_{-\infty}^{\infty} \tilde{h}(q_1, q_2) H_j(\sqrt{\omega}q_1) H_k(\sqrt{\omega}q_2) \exp\left\{-\frac{\omega}{2}(q_1^2 + q_2^2)\right\} \mathrm{d}q_1\mathrm{d}q_2\,. \tag{2.20}$$

3. Solution of the Cauchy problem

Define the operators $A = -\mathrm{i}K_+$, $B = -\mathrm{i}K_-$ and $C = K_3$. Then, from (2.3), we have

$$[C, A] = A \qquad [C, B] = -B \qquad \text{and} \qquad [A, B] = 2C\,. \tag{3.1}$$

Thus the operators A, B, C span a three-dimensional Lie-algebra which can be identified with $sl(2)$, the Lie-algebra of the special linear group $SL(2)$, (see [6]). A basis of $sl(2)$, which corresponds to A, B, C satisfying (3.1) is given by

$$J^+ = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \qquad J^- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \qquad J_3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}. \tag{3.2}$$

The actions of the 1-parameter subgroups $\exp tA, \exp tB$ and $\exp tC$ on a function $f(q_1, q_2)$ are obtained by solving the problems

$$\frac{\partial F}{\partial t} = AF \qquad F(q_1, q_2, 0) = f(q_1, q_2)$$
$$\frac{\partial F}{\partial t} = BF \qquad F(q_1, q_2, 0) = f(q_1, q_2)$$
$$\frac{\partial F}{\partial t} = CF \qquad F(q_1, q_2, 0) = f(q_1, q_2)$$

which are precisely the problems (2.7), (2.8) and (2.9). Thus we have

$$(\exp tA) f(q_1, q_2) = \sum_{m,n=-\infty}^{\infty} a_{mn} \exp\left[\frac{-\mathrm{i}}{4}(n^2 - m^2)t + \frac{\omega}{2}(q_1^2 + q_2^2) + \mathrm{i}\sqrt{\frac{\omega}{2}}((m+n)q_1 + (m-n)q_2)\right] \tag{3.3}$$

$$(\exp tB) f(q_1, q_2) = \sum_{r,s=-\infty}^{\infty} b_{rs} \exp\left[\frac{-\mathrm{i}}{4}(s^2 - r^2)t - \frac{\omega}{2}(q_1^2 + q_2^2) + \mathrm{i}\sqrt{\frac{\omega}{2}}((r+s)q_1 + (r-s)q_2)\right] \tag{3.4}$$

$$(\exp tC) f(q_1, q_2) = \sum_{j,k=0}^{\infty} c_{jk} H_j(\sqrt{\omega}q_1) H_k(\sqrt{\omega}q_2) \times \exp\left[-\frac{\omega}{2}(q_1^2 + q_2^2) - \frac{1}{2}(j + k + 1)t\right] \tag{3.5}$$

where the coefficients a_{mn}, b_{rs} and c_{jk} are given by (2.14), (2.17) and (2.20), respectively, on replacing $\tilde{f}, \tilde{g}$ and $\tilde{h}$ by f.

The matrix image of the partial differential equation (1.4) is

$$\frac{\partial \varphi}{\partial t} = \begin{pmatrix} -\mathrm{i}\omega & -\lambda \\ -\lambda & \mathrm{i}\omega \end{pmatrix} \varphi(t)\,. \tag{3.6}$$

The solution of (3.6) is easily seen to be

$$\varphi(t) = \begin{pmatrix} \cos\kappa t - \mathrm{i}(\omega/\kappa)\sin\kappa t & -(\lambda/\kappa)\sin\kappa t \\ -(\lambda/\kappa)\sin\kappa t & \cos\kappa t + \mathrm{i}(\omega/\kappa)\sin\kappa t \end{pmatrix} \varphi(0) \tag{3.7}$$

where $\kappa = \sqrt{\omega^2 - \lambda^2}$.

The solution of the Cauchy problems (1.4) can now be expressed as

$$\psi(q_1, q_2, t) = \mathrm{e}^{-\mathrm{i}tH}\tilde{\psi}(q_1, q_2)\,. \tag{3.8}$$

Since the operator H belongs to the Lie-algebra spanned by A, B, C, which is identified with $sl(2)$, the operator $\mathrm{e}^{-\mathrm{i}tH}$ can be expressed as

$$\mathrm{e}^{-\mathrm{i}tH} = \mathrm{e}^{\alpha(t)C}\mathrm{e}^{\beta(t)B}\mathrm{e}^{\gamma(t)A} \tag{3.9}$$

where the functions $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ can be computed using (3.2) and (3.7):

$$\alpha(t) = 2\log\left[\cos\kappa t - \frac{\mathrm{i}\omega}{\kappa}\sin\kappa t\right] \tag{3.10}$$

$$\beta(t) = \frac{\lambda}{\kappa}\sin\kappa t\left[\cos\kappa t - \frac{\mathrm{i}\omega}{\kappa}\sin\kappa t\right] \tag{3.11}$$

$$\gamma(t) = \frac{\lambda}{\kappa}\sin\kappa t\left[\cos\kappa t - \frac{\mathrm{i}\omega}{\kappa}\sin\kappa t\right]^{-1}. \tag{3.12}$$

Finally using the identities (3.3), (3.4), (3.5) and (3.9) in (3.8), we get the solution of the problem (1.4) as

$$\begin{aligned}\psi(q_1, q_2, t) = \sum_{j,k=0}^{\infty}\sum_{r,s=-\infty}^{\infty}\sum_{m,n=-\infty}^{\infty} & A_{mn}B_{mnrs}C_{rsjk} \\ & \times \exp\left[-\frac{\mathrm{i}}{4}(n^2 - m^2)\gamma(t) - \frac{\mathrm{i}}{4}(s^2 - r^2)\beta(t) - \frac{1}{2}(j + k + 1)\alpha(t)\right] \\ & \times H_j(\sqrt{\omega}q_1)H_k(\sqrt{\omega}q_2)\exp\left[-\frac{\omega}{2}(q_1^2 + q_2^2)\right]\end{aligned} \tag{3.13}$$

where

$$\begin{aligned}A_{mn} = \frac{\omega}{4\pi^2}\int_{-\pi/\sqrt{2\omega}}^{\pi/\sqrt{2\omega}}\int_{-\pi/\sqrt{2\omega}}^{\pi/\sqrt{2\omega}} & \tilde{\psi}(q_1, q_2) \\ & \times \exp\left[-\frac{\omega}{2}(q_1^2 + q_2^2) + \mathrm{i}\sqrt{\frac{\omega}{2}}((m + n)q_1 + (m - n)q_2)\right]\mathrm{d}q_1\mathrm{d}q_2\end{aligned} \tag{3.14a}$$

$$\begin{aligned}B_{mnrs} = \omega/4\pi^2\int_{-\pi/\sqrt{2\omega}}^{\pi/\sqrt{2\omega}}\int_{-\pi/\sqrt{2\omega}}^{\pi/\sqrt{2\omega}} & \times \exp\left[\omega(q_1^2 + q_2^2)\right. \\ & \left. + \mathrm{i}\sqrt{\frac{\omega}{2}} \times ((m + n + r + s)q_1 + (m + r - n - s)q_2)\right]\mathrm{d}q_1\mathrm{d}q_2\end{aligned} \tag{3.14b}$$

$$\begin{aligned}C_{rsjk} = \frac{\omega}{\pi}\frac{1}{2^{j+k}j!k!}\int_{-\infty}^{\infty} & \exp\left[-\omega(q_1^2 + q_2^2) + \mathrm{i}\sqrt{\frac{\omega}{2}}((r + s)q_1 + (r - s)q_2)\right] \\ & \times H_j(\sqrt{\omega}q_1)H_k(\sqrt{\omega}q_2)\mathrm{d}q_1\mathrm{d}q_2\end{aligned} \tag{3.14c}$$

and $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are given by (3.10), (3.11) and (3.12), respectively.

The algebraic technique which we have used in the present work provided us with a powerful and feasible tool to solve a wide class of partial differential equation, and it can be regarded as a useful alternative to more classical methods that do not provide the same generality. To see that, let us turn our attention to find the solution of the Schrödinger equation (1.4) by employing the classical method. To do so we shall use the definition given by (2.4) together with (1.3) and (1.4), then we have

$$\frac{\partial^2\psi}{\partial q_1^2}+\frac{\partial^2\psi}{\partial q_2^2}-\omega^2(q_1^2+q_1^2)\psi-2\lambda\left[\omega q_1q_2+\frac{1}{\omega}\frac{\partial^2}{\partial q_1\partial q_2}\right]\psi=-2\mathrm{i}\frac{\partial\psi}{\partial t}. \tag{3.15}$$

By changing the variables q_1 and q_2 according to (2.10) we find

$$(\omega+\lambda)\frac{\partial^2\bar\psi}{\partial x^2}+(\omega-\lambda)\frac{\partial^2\bar\psi}{\partial y^2}-(\omega-\lambda)x^2\bar\psi-(\omega+\lambda)y^2\bar\psi-(\omega+\lambda)y^2\bar\psi=-2\mathrm{i}\frac{\partial\bar\psi}{\partial t}. \tag{3.16}$$

Under the condition $\omega>\lambda$ (physical condition), we can express the general solution in the form

$$\bar\psi(x,y,t)=NH_{n'}\left[\sqrt{\frac{\omega-\lambda}{k}}x\right]H_{m'}\left[\sqrt{\frac{\omega+\lambda}{k}}y\right]\exp\left(-\frac{1}{2k}\left[(\omega-\lambda)x^2+(\omega+\lambda)y^2\right]\right)$$
$$\times\exp[-\mathrm{i}(n'+m'+1)kt] \tag{3.17}$$

where N is the normalizing constant, and k is given by (3.7). On reverting to physical quantities and calculating the normalizing constant we have the following expression

$$\psi(q_1,q_2,t)=\sum_{m',n'=0}^{\infty}NH_{n'}\left[\sqrt{\frac{\omega^2-\lambda\omega}{2k}}(q_1-q_2)\right]H_{m'}\left[\sqrt{\frac{\omega^2+\lambda\omega}{2k}}(q_1+q_2)\right]$$
$$\times\exp\left(-1/2k\left[\omega^2(q_1^2+q_2^2)+2\lambda\omega q_1q_2\right]\right)\exp\left[-\mathrm{i}k(n'+m'+1)t\right] \tag{3.18}$$

where

$$N=\left(\frac{\omega}{\pi}\right)(n'!m'!)^{-1}2^{-(m'+n')}\int_{-\infty}^{\infty}\bar\psi(q_1,q_2)H_{n'}\left[\sqrt{\frac{\omega^2-\lambda\omega}{2k}}(q_1-q_2)\right]$$
$$\times H_{m'}\left[\sqrt{\frac{\omega^2+\lambda\omega}{2k}}(q_1+q_2)\right]$$
$$\times\exp\left(-\frac{1}{2k}\left[\omega^2(q_1^2+q_2^2)+2\lambda\omega q_1q_2\right]\right)\mathrm{d}q_1\mathrm{d}q_2. \tag{3.19}$$

Now, if we make a comparison between the two solutions in (3.13) and (3.18), we may say that, although the method adopted to derive (3.13) is more complicated than that in (3.18), it has the advantage of including all the possibilities for the Schrödinger wavefunction of the parametric amplifier. Furthermore, it avoids all the singularities which would appear in (3.18) if we did not restrict ourselves with the condition $\omega>\lambda$.

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